

FINITE DYNAMICAL SYSTEMS, LINEAR AUTOMATA, AND FINITE FIELDS

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ABSTRACT. We establish a connection between finite fields and finite dynamical systems. We show how this connection can be used to shed light on some problems in finite dynamical systems and in particular, in linear systems.

1. INTRODUCTION

There is a natural correspondence between the set $GF(p^r)$ and the set \mathbf{Z}_p^r of r -tuples over \mathbf{Z}_p , p prime. Furthermore, $GF(p^r)$ is a vector space over $GF(p) = \mathbf{Z}_p$ and linear transformations over \mathbf{Z}_p^r correspond to linearized polynomials over $GF(p^r)$. In this ongoing work, we use these facts to study some problems in finite dynamical systems and linear automata.

In Section 2, we study finite dynamical systems and how our approach can be used in the classification problem. In Section 3, we study linearized polynomials and how they can be applied to a problem in linear finite state machines, which in turn arises from a problem in crystallographic FFTs.

2. FINITE DYNAMICAL SYSTEMS

Finite dynamical systems are important in applications to computational molecular biology. They are useful in microarrays of genes in order to find the best model that fits a given data, the so called “reverse engineering problem.” (See <http://industry.ebi.ac.uk/~brazama/Genenets>.) Laubenbacher and Pareigis [2] define a finite dynamical system as a function $f : k^n \rightarrow k^n$, constructed by the following data:

1. $k = \{0, 1\}$
2. a finite graph F on n vertices.
3. a family of “local” update functions $f_a : k^n \rightarrow k^n$, one for each vertex $a \in F$, which changes only the coordinate corresponding to a , and computes the binary state of vertex a . These functions are local in the sense that they only depend on those variables which are connected to $a \in F$.
4. an “update schedule” π , which specifies an order on the vertices of F , represented by a permutation $\pi \in S_n$.

The function f is then constructed by composing the local functions according to the update schedule π , that is

$$f = f_{\pi(n)} \circ \cdots \circ f_{\pi(1)} : k^n \rightarrow k^n.$$

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In [3], Laubenbacher and Pareigis called the above function a permutation sequential dynamical system, and extended the definition in different directions. In particular, they take k as an arbitrary set. For our purposes it is convenient to consider $k = \mathbf{Z}_m = \{0, 1, \dots, m-1\}$,

Definition 2.1. A finite dynamical system (FDS) is a pair (V, f) where V is the set of vectors over a finite field and $f : V \rightarrow V$.

Definition 2.2. The state diagram of a FDS (V, f) is the digraph whose vertices are members of V and whose edges are the set of all $(x, f(x))$, where $x \in V$.

Remark: Note how for an FDS (\mathbf{Z}_p^r, f) , the function can be viewed naturally as an r -tuple (f_1, f_2, \dots, f_r) of functions $f_i : \mathbf{Z}_p \rightarrow \mathbf{Z}_p$, $i \leq i \leq r$. It is also important to note that, using Langrange interpolation, any function from a finite field to itself can be realized as a polynomial [4]. Hence each of the f_i can be regarded as a polynomial over \mathbf{Z}_p . A similar remark applies to a FDS of the form $(GF(p^n), f)$.

Definition 2.3. Two FDSs are isomorphic if their state diagrams are isomorphic.

Two FDSs are isomorphic if and only if their state diagram are isomorphic as digraphs. A *limit cycle* is simply a directed cycle in the state diagram \mathcal{S}_f . A loop is a limit cycle consisting of a single vertex and in the case that it occurs, it is a fixed point of the FDS f . We denote by \mathcal{L}_f the subdigraph of \mathcal{S}_f induced by all the arcs of the limit cycles.

Definition 2.4. Let $f : V \rightarrow V$ be a FDS with state diagram \mathcal{S}_f and with subdigraph \mathcal{L}_f of limit cycles. Then $x \in k$ is a vertex in \mathcal{L}_f if and only if there exists a positive integer m such that $f^m(x) = x$. The minimum m such that $f^m(x) = x$ for all $x \in \mathcal{L}_f$ is called the *order* of the system f , denoted by $\text{Order}(f)$. (See [2])

Directed paths in \mathcal{S}_f correspond to iterations of f on the element at the beginning of the path. Since the set \mathbf{Z}_p^r is finite, any directed path must eventually enter a limit cycle. Thus each connected component of \mathcal{S}_f consist of one limit cycle, together with *transients*, that is, directed paths having no repeated vertices and ending in a vertex that is part of limit cycle.

One of the main problems in FDSs is their classification. Loosely speaking, this is the problem of determining of two arbitrarily given FDSs whether or not they are isomorphic. One of our goals in this work is to facilitate the solution of the classification problem though the association given in

Theorem 2.5. For any fixed basis $\alpha_1, \dots, \alpha_r$ of $GF(p^r)$ there is a natural one-one correspondence between the FDSs over $GF(p^r)$ and those over \mathbf{Z}_p^r .

Proof. There is a natural correspondence between the sets \mathbf{Z}_p^r and $GF(p^r)$, namely, $(x_1, x_2, \dots, x_r) \leftrightarrow x_1\alpha_1 + \dots + x_r\alpha_r$. Now given $f : \mathbf{Z}_p^r \rightarrow \mathbf{Z}_p^r$, define $L : GF(p^r) \rightarrow GF(p^r)$ such that for each (x_1, x_2, \dots, x_r) in \mathbf{Z}_p^r , $L(x) = f_1(x_1)\alpha_1 + \dots + f_r(x_r)\alpha_r$, where $x = x_1\alpha_1 + \dots + x_r\alpha_r$ and $f = (f_1, \dots, f_r)$. Conversely, given $L : GF(p^r) \rightarrow GF(p^r)$, if $L(x_1\alpha_1 + \dots + x_r\alpha_r) = y_1\alpha_1 + \dots + y_r\alpha_r$, there corresponds a function $f = (f_1, f_2, \dots, f_r)$ such that $f(x_1, \dots, x_r) = (y_1, \dots, y_r)$, where $f_i(x_i) = y_i$ for each $i = 1, 2, \dots, r$. Since this correspondence is onto and the two sets are finite with the same number of elements, it is also one-one.

Corollary 2.6. If $S_1 = (\mathbf{Z}_p^r, f)$ and $S_2 = (\mathbf{Z}_p^r, f')$ are FDSs and f corresponds to $L(x)$ with respect to the basis $\alpha_1, \dots, \alpha_r$ and f' corresponds also to $L(x)$, but with respect to another basis, then S_1 and S_2 are isomorphic.

This latter corollary says that our approach is quite useful for the classification problem. On the other hand,

Theorem 2.7. For any fixed basis $\alpha_1, \dots, \alpha_r$ of $GF(p^r)$, there is a natural correspondence between the FDSs over $(GF(p^r))^n$ and those over $(\mathbf{Z}_p)^{rn}$.

In other words, it is redundant to study both types of these FDSs, but each is important given the classification problem.

3. LINEAR FDSs AND LINEARIZED POLYNOMIALS.

A linear finite dynamical system or a linear (autonomous) finite state machine is a FDS (\mathbf{Z}_p^r, f) in which f is a linear transformation on \mathbf{Z}_p^r regarded as a vector space over \mathbf{Z}_p . We shall see in this case that there is a useful correspondence between linear FDSs and linearized polynomials.

The correspondence $x \rightarrow x^{p^i}$, $i = 0, 1, \dots, p^r-1$, gives the Galois automorphisms of $GF(p^r)$. A linearized polynomial $L(x)$ is a polynomial generated by these automorphisms. In other words, $L(x) = \sum_{i=0}^{r-1} A_i x^{p^i}$, where $A_i \in GF(p^r)$. We note that if $y, z \in GF(p^r)$ and $\lambda \in GF(p)$, then $L(x+y) = L(x) + L(y)$ and $L(\lambda x) = \lambda L(x)$. Thus, $L(x)$ is a linear function on $GF(p^r)$ regarded as a vector space over $GF(p)$. Furthermore the correspondence between $GF(p^r)$ and \mathbf{Z}_p^r given in Theorem 1 is an isomorphism as a vector space over \mathbf{Z}_p . Since there are $(p^r)^r$ linearized polynomials, this coincides with all the linear functions on \mathbf{Z}_p^r . It is easy to see that if $f : \mathbf{Z}_p^r \rightarrow \mathbf{Z}_p^r$ is a FDS associated to the linearized polynomial $L(x) = \sum_{i=0}^{r-1} A_i x^{p^i}$, then $\ker f$ is the set of all roots of $L(x)$. So, f is invertible if and only if the only root of L in $GF(p^r)$ is 0.

Given a linear autonomous machines $S = (\mathbf{Z}_p^r, F)$, if f is a nonsingular linear transformation, i.e., an invertible matrix over \mathbf{Z}_p , then the state space \mathbf{Z}_p^r decomposes into disjoint “orbits” or “cycles.” Based on the above observation, the same also holds for machines $(GF(p^r), L)$ and it is interesting to note how properties of linearized polynomials determine this orbit structure, much in the same way as the properties of the elementary divisors of f determine the orbit structure of (\mathbf{Z}_p^r, f) in the classical theory. However, our motivation for studying linearized polynomials stems from a more general problem which arises in crystallographic FFTs [5]. Let us briefly describe this problem.

Crystallographic data can introduce structured symmetries in the inputs of a multidimensional discrete Fourier transform, which in turn introduce symmetries into the outputs. In order to avoid redundant calculations, it is of interest to exploit these symmetries. Assuming that symmetries are given by an $n \times n$ matrix S over \mathbf{Z}_p , for prime p edge length, we can reduce the complexity of the FFT by determining a matrix M with $MS = SM$ and $M^t S = SM^t$ (where M^t denotes the transpose of M) that minimizes the number of “ MS -orbits.” A vector $x \in \mathbf{Z}_p^n$ belongs to an MS -orbit of length k if and only if $M^k x = S^i x$ for some i . The cases $n = 2$ and $n = 3$ are of particular interest.

For $n = 2$, for example, i.e., $F : \mathbf{Z}_p \times \mathbf{Z}_p \rightarrow \mathbf{Z}_p \times \mathbf{Z}_p$, this corresponds to the study of linearized polynomials $F(x) = Ax^p + Bx$, where $A, B \in GF(p^2)$, where F is an invertible map. The first question is, when is $F(x)$ invertible in $GF(p^2)$?

Lemma 3.1. $F(x)$ is invertible if $A^{p+1} \neq B^{p+1}$.

Proof. $F(x)$ is invertible if and only if it is one-one, i.e., if $\ker F = 0$. In other words, if $F(x) = 0$ has only $x = 0$ as a solution over $GF(p^2)$. But $x \neq 0$ and $Ax^p + Bx = 0$ imply that $X^{p-1} = -\frac{B}{A}$ and so raising both sides to the power $p+1$, we obtain $x^{p^2-1} = \frac{B^{p+1}}{A^{p+1}}$ and $A^{p+1} = B^{p+1}$.

In the remainder of this section we consider the class \mathcal{L}_p of linearized polynomials $L(x) = \sum_{i=0}^{r-1} A_i x^{p^i}$ where $A_i \in GF(p)$. These types of polynomials have important properties which we outline below.

Property I. If $L(x), L'(x) \in \mathcal{L}_p$ then $L(L'(x)) = L'(L(x))$. (Note that this means that the corresponding matrices commute).

Property II. Given $L(x) \in \mathcal{L}_p$, the class of $L'(x)$ satisfying Property I is precisely \mathcal{L}_p .

Property III. Using a normal basis, the matrix corresponding to $L(x) \in \mathcal{L}_p$ is symmetric. Consequently, the transpose matrix also commutes.

Definition 3.2. If $L(x) = \sum_{i=0}^{r-1} A_i x^{p^i}$ is a linearized polynomial, then its *associate* is $l(x) = \sum_{i=0}^{r-1} A_i X^i$.

Definition 3.3. $L^i(x) = L(x)$ and $L^{n+1}(x) = L(L^n(x))$. That is, $L^n(x)$ is the n -fold composition of L with itself.

Property IV. $L^n(x) = x$ modulo $x^{p^r} = x$ if and only if $(l(x))^n = 1$ modulo $x^r = 1$.

4. SYSTEMS OVER \mathbf{Z}_{p^n}

In this section we study FDS over \mathbf{Z}_{p^n} , that is systems (V, f) where $f : \mathbf{Z}_{p^n}^r \rightarrow \mathbf{Z}_{p^n}^r$ and V is the \mathbf{Z}_{p^n} -module $\mathbf{Z}_{p^n}^r$. We use Definitions 2.2, 2.3, and 2.4, but now we are working in the \mathbf{Z}_{p^n} -module V . Therefore a linear FDS is an endomorphism of the \mathbf{Z}_{p^n} -module V .

Let $g : \mathbf{Z}_{p^n}^r \rightarrow \mathbf{Z}_{p^n}^r$ be a bijection. Then the product function $g^r : \mathbf{Z}_{p^n}^{nr} \rightarrow \mathbf{Z}_{p^n}^{nr}$ given by $g(a_1, a_2, \dots, a_r) = (g(a_1), g(a_2), \dots, g(a_r))$ is a bijection too.

Proposition 4.1. Let $f : \mathbf{Z}_{p^n}^r \rightarrow \mathbf{Z}_{p^n}^r$ be a FDS. Let $\bar{f} : \mathbf{Z}_{p^n}^{nr} \rightarrow \mathbf{Z}_{p^n}^{nr}$ be the FDS such that $g^r \circ \bar{f} = f \circ g^r$. Then \bar{f} and f have state diagrams isomorphic.

Proof. Since g^r is a bijection, there exists $(g^r)^{-1} = g^{-r}$. Then the system $f_1 = g^{-r} \circ f \circ g^r$ has the same state diagram to f . In fact, set $x_1 = g^{-r}(x)$ and $y_1 = g^{-r}(y) = (g^{-r} \circ f \circ g^r)(x_1)$. Now, suppose $(x, y = f(x))$ is an edge in the state diagram of f . Then (x_1, y_1) is an edge in the state diagram of f_1 . On the other hand, $g^r \circ f_1 = f \circ g^r$. So, $\bar{f} = f_1$ and our claim holds.

Definition 4.2. With the notation above, if f is a linear FDS over \mathbf{Z}_{p^n} then the system f will be called the linear system associated to \bar{f} by the bijection g .

We use the the linear system associated to a non-linear FDS \bar{f} to describe its state diagram and its order.

Example

Let $f : (\mathbf{Z}_{2^3})^2 \rightarrow (\mathbf{Z}_{2^3})^2$ be a linear system, given by $f(a, b) = (5b, a + 2b)$. For any bijection $g : (\mathbf{Z}_2)^3 \rightarrow \mathbf{Z}_{2^3}$ we have an induced systems \bar{f} with the same state

diagram of the system f . Now using the method given in [1], we find the Order of f and the Order of \bar{f} for any bijection g . The matrix

$$A = \begin{pmatrix} 0 & 5 \\ 1 & 2 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2}$$

has minimal polynomial

$$m(x) = (x - 1)^2$$

and the Order of A modulo 2 is $e = 2$. Since

$$A^2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \pmod{2^3},$$

the largest positive integer β such that $A^2 \equiv I \pmod{2^\beta}$ is $\beta = 1$. Then A^2 has Order 4 and A has Order 8 modulo 2^3 .

5. FUTURE WORK

We will exploit the ideas presented here to seek a polynomial solution to the reverse engineering problem. We also make use of our theory to develop an efficient algorithm to determine, given a matrix S of symmetries, a matrix M that minimizes the number of MS -orbits in the precomputation phase of crystallographic FFTs.

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